Hamilton

1. A number of couples met and each person shook hands with everyone else present, but not with themselves or their partners.

There were 31 000 handshakes altogether.

How many couples were there?

Solution

Suppose that there were c couples. Then each couple interacted with $c - 1$ other couples (every couple except themselves).

Then $c \times (c - 1)$ gives double the total number of interactions between couples (since we have counted each interaction exactly twice – once for each couple involved in it). So the total number of interactions is $\frac{1}{2}c(c - 1)$.

Each interaction between couples results in four handshakes, so the total number of handshakes was $4 \times \frac{1}{2}c(c - 1)$, that is, $2c(c - 1)$.

Therefore $2c(c - 1) = 31\,000$, so that $c^2 - c - 15\,500 = 0$. Hence $(c - 125)(c + 124) = 0$, so that $c = 125$ (ignoring the negative answer, which is not practicable here).

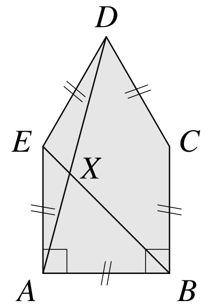
Thus there were 125 couples.

Note

If you have not yet seen how to solve a quadratic equation, you could also notice that you are looking for two consecutive numbers which multiply to 15 500. Since one of these needs to be a multiple of 25 (can you see why?), completing a methodical search is not too onerous and, if explained thoroughly, is just as valid a solution.

2. The diagram shows a pentagon $ABCDE$ in which all sides are equal in length and two adjacent interior angles are 90°. The point X is the point of intersection of AD and BE.

Prove that $DX = BX$.



Solution

AE and BC are equal, and are parallel because the angles at A and B are both equal to 90° (allied angles, converse). Hence $ABCE$ is a parallelogram (opposite sides equal and parallel), so that EC and AB are equal (property of a parallelogram).

It follows that $ABCE$ is a square (a rhombus with a right angle), and that triangle CDE is equilateral (all sides equal).

Now triangle ADE is isosceles, so that $\angle ADE = \angle DAE$ (base angles of an isosceles triangle). However, $\angle DEA = 60° + 90° = 150°$, so that $\angle ADE = \angle DAE = 15°$ (angle sum of a triangle). Similarly, triangle BCD is isosceles, and$\angle CDB = \angle DBC = 15°$.

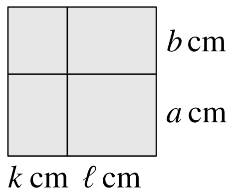
Therefore $\angle BDX = 60° - 15° - 15° = 30°$.

Also $\angle XBC = 45°$ (a diagonal of a square bisects the angle), so that $\angle XBD = 45° - 15° = 30°$.

Hence $\angle XBD = \angle BDX$, so $DX = BX$ (sides opposite equal angles).

> **3.** A 4 cm × 4 cm square is split into four rectangular regions using two line segments parallel to the sides.
>
> How many ways are there to do this so that each region has an area equal to an integer number of square centimetres?

SOLUTION



Let the lengths of the sides of the regions be k cm, ℓ cm, a cm and b cm, as shown. We know that each of these is less than 4 cm.

Then the regions have areas $ka\,\text{cm}^2$, $kb\,\text{cm}^2$, $\ell a\,\text{cm}^2$ and $\ell b\,\text{cm}^2$ respectively, and thus each of ka, kb, ℓa and ℓb is an integer.

Hence $ka + kb = k(a + b) = 4k$ is an integer, so that k is an integer multiple of $\frac{1}{4}$. Similarly, each of ℓ, a and b is an integer multiple of $\frac{1}{4}$.

However, none of k, ℓ, a and b is an integer multiple of 4, because each is between 0 and 4. Hence each of them has to be an integer multiple of $\frac{1}{2}$ in order that each of ka, kb, ℓa and ℓb is an integer.

If both a and k are integers

then clearly all four rectangles have integer area. In this case we have three choices for a (1, 2 and 3) and three choices for k (also 1, 2 and 3), giving $3 \times 3 = 9$ choices.

If one of a or k is not an integer (that is, an odd multiple of $\frac{1}{2}$)

then the other is 2, otherwise ka is not an integer. When $a = 2$ it is possible for k to take four such values $\left(\frac{1}{2}, \frac{3}{2}, \frac{5}{2} \text{ and } \frac{7}{2}\right)$ and similarly when $k = 2$, a could take the same four values.

Note that once a and k have been chosen, then these determine b and ℓ uniquely.

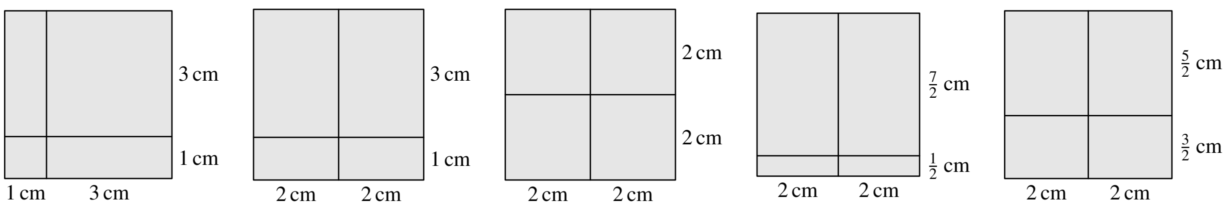
Thus altogether there are $9 + 4 + 4 = 17$ ways.

NOTE

If the grid can be rotated or reflected, so that, for example, $a = 1$ is counted as being the same as $k = 1$ or $a = 3$, then there are 5 ways, given by the following pairs of values of (a, k) or (k, a):

$$(1, 1), (1, 2), (2, 2), \left(\tfrac{1}{2}, 2\right), \left(\tfrac{3}{2}, 2\right),$$

as shown below.



4. Each of A and B is a four-digit palindromic integer, C is a three-digit palindromic integer, and $A - B = C$.

What are the possible values of C?

[*A palindromic integer reads the same 'forwards' and 'backwards'.*]

Solution

Let the integers be A = '$adda$', B = '$beeb$' and C = 'cfc'. Then we may rewrite $A - B = C$ in the form of the addition sum shown.

$$\begin{array}{r} b\,e\,e\,b \\ +\,c\,f\,c \\ \hline a\,d\,d\,a \end{array}$$

Note that none of a, b and c can be zero, because A, B and C are integers with 4, 4 and 3 digits.

Considering the 'ones' column, $a \neq b$ since $c \neq 0$. Considering the 'thousands' column, since $a \neq b$, there is a 'carry' from the 'hundreds' column. It follows that $a = b + 1$, so that, from the 'ones' column, $c = 1$. Once we know that $c = 1$, the only way there can be a 'carry' from the 'hundreds' to the 'thousands' column is to have $e = 9$ or $e = 8$.

There are two cases.

***e* = 9**

In this case, we have the addition shown alongside.

$$\begin{array}{r} b\,9\,9\,b \\ +\,1\,f\,1 \\ \hline a\,d\,d\,a \end{array}$$

Considering the 'hundreds' column, d is either 0 or 1, depending on whether there is not, or is, a 'carry' from the 'tens' column. But when $d = 0$ there *is* such a 'carry' ($f = 1$), which leads to a contradiction.

Hence $d = 1$ and thus $f = 2$, as shown alongside.

$$\begin{array}{r} b\,9\,9\,b \\ +\,1\,2\,1 \\ \hline a\,1\,1\,a \end{array}$$

***e* = 8**

We now have the addition shown alongside.

$$\begin{array}{r} b\,8\,8\,b \\ +\,1\,f\,1 \\ \hline a\,d\,d\,a \end{array}$$

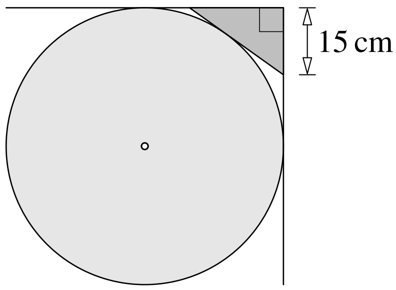
In this case, we similarly have $d = 0$ and $f = 2$, as shown alongside.

$$\begin{array}{r} b\,8\,8\,b \\ +\,1\,2\,1 \\ \hline a\,0\,0\,a \end{array}$$

Therefore, in either case, there is only one possible value of C, namely 121. An important step is to check whether it is actually possible to find values of A and B which give $C = 121$, and working on from one of the above cases, it is easy to find an example, such as $2112 - 1991 = 121$.

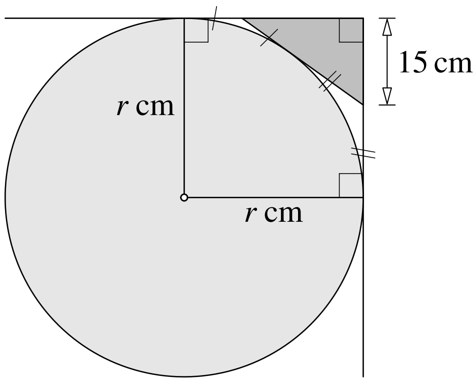
5. The area of the right-angled triangle in the diagram alongside is $60\,\text{cm}^2$. The triangle touches the circle, and one side of the triangle has length 15 cm, as shown.

What is the radius of the circle?



Solution

The area of a triangle is $\frac{1}{2}$base × height, so the triangle has sides of length 8 cm, 15 cm and 17 cm (using Pythagoras' theorem).



Let the radius of the circle be r cm.

Draw two radii, as shown. Since a tangent meets the radius at the point of contact at right angles, we have a square, with sides of length r cm.

Then, using the equal tangent theorem, we have

$$(r - 8) + (r - 15) = 17,$$

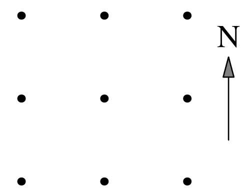
so that $2r = 40$ and $r = 20$.

Commentary

The equal tangent theorem states that the two tangents from a point outside a circle to the circle are equal in length. Here we use two of the vertices of the triangle as the points.

Hence the radius of the circle is 20 cm.

6. Nine dots are arranged in the 2×2 square grid shown. The arrow points north.



Harry and Victoria take it in turns to draw a unit line segment to join two dots in the grid.

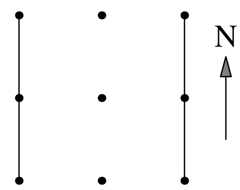
Harry is only allowed to draw an east-west line segment, and Victoria is only allowed to draw a north-south line segment. Harry goes first.

A point is scored when a player draws a line segment that completes a 1×1 square on the grid.

Can either player force a win, no matter how the other person plays?

Solution

Victoria can force a win, by adopting the following strategy.



She plays her first four moves on the sides of the grid, as shown alongside (where only Victoria's moves are shown).

Victoria then plays to complete 1×1 squares.

Why does this strategy work?

If Victoria plays in this way, then Harry is unable to score with his first five moves (since there are no squares with their two north-south edges drawn whenever he plays, so he cannot possibly draw the fourth edge of a square).

After Harry has played for the fifth time, he will have drawn both east-west edges of at least one of the 1×1 squares. Victoria can then draw one of the central two north-south lines to complete at least one square and so score.

When Harry plays his sixth (and final) move, he has no choice where to play. This move may complete a square and win Harry a point (but cannot complete more than one, since one of the central north-south lines remains undrawn so there are two squares which Harry cannot possibly complete yet).

Once Harry has drawn his final segment, there are two adjacent 1×1 squares which share a common undrawn edge, and all other squares have been completed. On her sixth move, therefore, Victoria can draw the final remaining undrawn line segment, completing these final two squares.

Since Harry has scored at most one point and Victoria has scored on two separate occasions, her points total is higher than his, and so she wins.

Note

There are two possible ways to interpret the scoring, depending on whether one or two points are awarded for completing *two* 1×1 squares in the same move. The strategy we give works in either case.

